FUNCTIONS OF BOUNDED BOUNDARY ROTATION

BY

BERNARD PINCHUK

ABSTRACT

Classes of functions U_k, which generalize starlike functions in the same manner that the class V_k of functions with boundary rotation bounded by $k\pi$ generalizes convex functions, are defined. The radius of univalence and starlikeness is determined. The behavior of $f_\alpha(z) = \int_0^z [f'(t)]^\alpha dt$ is determined for various classes of functions. It is shown that the image of $|z| < 1$ under V_k functions contains the disc of radius $1/k$ centered at the origin, and V_k functions are continuous in $|z| \leqq 1$ with the exception of at most $[k/2 + 1]$ points on $|z| = 1$.

1. All functions $f(z)$ are analytic in the open unit disc D and are normalized by the conditions $f(0) = 0$, $f'(0) = 1$, unless stated otherwise.

$\boldsymbol{P}$ will denote the class of functions $P(z)$ which are analytic, have positive real part in D, and satisfy $P(0) = 1$.

M_k will denote the class of real-valued functions $m(t)$ of bounded variation on $[-\pi, \pi]$ which satisfy the conditions

$$\int_{-\pi}^{\pi} dm(t) = 2, \quad \int_{-\pi}^{\pi} |dm(t)| \leqq k$$

M_2 is clearly the class of nondecreasing functions on $[-\pi, \pi]$ satisfying $\int_{-\pi}^{\pi} dm(t) = 2$.

If $m(t) \in M_k$ with $k > 2$ we can write $m(t) = \alpha(t) - \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are both nondecreasing functions on $[-\pi, \pi]$ and satisfy

$$\int_{-\pi}^{\pi} d\alpha(t) \leqq \frac{k}{2} + 1 \quad \text{and} \quad \int_{-\pi}^{\pi} d\beta(t) \leqq \frac{k}{2} - 1.$$

DEFINITION 1.1. *A function $f(z)$ is said to be in the class V_k if*

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(1.1) $$f'(z) = \exp \int_{-\pi}^{\pi} -\log(1 - ze^{-it})\, dm(t),$$

for some $m(t) \in M_k$.

$V_2 = C$ is the class of convex univalent functions in D.

S^* will denote the class of starlike univalent functions in D.

It is known that $f(z) \in S^*$ if and only if

(1.2) $$f(z) = z \exp \int_{-\pi}^{\pi} -\log(1 - ze^{-it})\, dm(t)$$

for some $m(t) \in M_2$.

Comparing (1.1) with (1.2) we have the well-known relation

(1.3) $$f(z) \in C \text{ iff } zf'(z) \in S^*.$$

2. As is clear from the discussion above, the class V_k is a generalization of C. Indeed, if we begin with the representation (1.1) for C by restricting k to be 2, we then obtain the classes V_k by allowing the function $m(t)$ to range over M_k for a given $k > 2$.

We now generalize the class S^* in the same way, namely, by allowing $m(t)$ to range over the class M_k in (1.2).

DEFINITION 2.1. *A function $f(z)$ is said to be in the class U_k if*

(2.1) $$f(z) = z \exp \int_{-\pi}^{\pi} -\log(1 - ze^{-it}) dm(t)$$

for some $m(t) \in M_k$.

We have in analogy with (1.3),

PROPOSITION 2.1. $f(z) \in V_k$ *iff* $zf'(z) \in U_k$.

Equivalently,

$$f(z) \in U_k \text{ iff } \int_0^z \frac{f(\zeta)}{\zeta} d\zeta \in V_k.$$

Before proceeding with our discussion of U_k, we present a definition and a lemma which will simplify matters.

DEFINITION 2.2. $\boldsymbol{P_k}$ *denotes the class of functions which are analytic in D and have the representation*

$$P(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,dm(t) \tag{2.2}$$

where $m(t) \in M_k$.

Clearly, $\boldsymbol{P}_2 = \boldsymbol{P}$.

LEMMA 2.1. *All functions in* $\boldsymbol{P}_k$ *have positive real part for*

$$|z| < \frac{k - \sqrt{k^2-4}}{2}.$$

Furthermore, there exist functions in $\boldsymbol{P}_k$ *which do not have positive real part in any larger disc.*

PROOF. Let $P(z) \in \boldsymbol{P}_k$ with $P(z)$ given by (2.2). Set $m(t) = \alpha(t) - \beta(t)$ as in Section 1, and assume that $\int_{-\pi}^{\pi} |dm(t)| = k$. Then, setting $z = re^{i\theta}$, we have

$$\begin{aligned} \operatorname{Re} P(z) &= \frac{1}{2}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - t)+r^2}\,[d\alpha(t) - d\beta(t)] \\ &\geqq \frac{1}{2}\left[\left(\frac{k}{2}+1\right)\frac{1-r^2}{1+2r+r^2} - \left(\frac{k}{2}-1\right)\frac{1-r^2}{1-2r+r^2}\right] \\ &= \frac{1}{1-r^2}(r^2 - kr + 1) \end{aligned}$$

From here it is clear that $\operatorname{Re} P(z) > 0$ for $|z| = r < \dfrac{k - \sqrt{k^2-4}}{2}$.

If $\int_{-\pi}^{\pi} |dm(t)| = k' < k$, we certainly have the result since in this case $\operatorname{Re} P(z) > 0$ for $|z| < \dfrac{k' - \sqrt{k'^2-4}}{2}$ which is a larger circle.

The function

$$P(z) = \frac{1}{2}\left[\left(\frac{k}{2}+1\right)\frac{1-z}{1+z} - \left(\frac{k}{z}-1\right)\frac{1+z}{1-z}\right] \tag{2.3}$$

shows that this is sharp.

We now return to the classes U_k. It is known [3] that $f(z) \in V_k$, $2 \leqq k \leqq 4$, implies that $f(z)$ is univalent (in fact, close-to-convex [4]) in D. This is not the case for U_k.

THEOREM 2.1. *All functions in* U_k *are starlike univalent for* $|z| < \dfrac{k - \sqrt{k^2-4}}{2}$.

Furthermore, there are functions in U_k which are not univalent in any larger disc.

PROOF. Let $f(z) \in U_k$ with $f(z)$ given by (2.1). A simple calculation shows that

$$\frac{zf'(z)}{f(z)} \in \boldsymbol{P}_k$$

Thus, $f(z)$ is starlike univalent for $|z| < \dfrac{k - \sqrt{k^2 - 4}}{2}$.

To prove the second assertion, consider the function

$$f(z) = \frac{z(1-z)^{k/2-1}}{(1+z)^{k/2+1}}.$$

This function is in U_k and has vanishing derivative at $z = \dfrac{k - \sqrt{k^2 - 4}}{2}$.

COROLLARY 2.1. *The radius of convexity for the class V_k is* $\dfrac{k - \sqrt{k^2 - 4}}{2}$.

PROOF. If $f(z) \in V_k$ then $1 + \dfrac{zf''(z)}{f'(z)} \in \boldsymbol{P}_k$ and the result follows from Lemma 2.1.

It is clear that the radius of convexity for V_k is, by definition, the same as the radius of starlikeness for U_k. Recently, M. S. Robertson [7] has obtained the radius of convexity for V_k, and thus one could obtain our theorem from that result.

In view of the relation between U_k and V_k as given in Proposition 2.1., it is an easy matter to write down the extremal functions in U_k for the most general extremal problems once the corresponding solutions are known in V_k. The reader is referred to [4] where extremal problems and rotation and distortion theorems are discussed for V_k. The corresponding results for U_k can be read off from that work, and we omit them here.

3. It is well known that the analytic functions in D whose derivatives lie in $\boldsymbol{P}$ are univalent in D. Let $\boldsymbol{B}_k$ denote the analytic functions in D whose derivations lie in $\boldsymbol{P}_k$. As a result of Lemma 2.1 we have

THEOREM 3.1. *The radius of univalence for $\boldsymbol{B}_k$ is* $\frac{1}{2}(k - \sqrt{k^2 - 4})$.

That this is sharp can be seen by considering the function $f(z) = \int_0^z P(\zeta)d\zeta$ where $P(z)$ is the function (2.3).

It is a simple procedure now to apply the Goluzin variation method to the classes $\boldsymbol{P}_k$ or $\boldsymbol{B}_k$ as was done for V_k in [4], in order to solve general extremal problems and to obtain rotation and distortion theorems. That would take us too far afield, and we omit the details.

4. We now return to Proposition 2.1 and ask the following question: Suppose $f(z) \in U_k$ and α is a given number, $0 < \alpha < 1$. What can we say about

$$\int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha d\zeta\ ?$$

This is equivalent to investigating the behavior of $f_\alpha(z) = \int_0^z (f'(\zeta))^\alpha d\zeta$ for $f(z) \in V_k$.

The corresponding question for the class S of normalized univalent functions in D has been studied in [1]. There the question is whether $f_\alpha(z)$ is univalent, and the answer is not yet known for all α in $0 < \alpha < 1$. This problem has also been studied for close-to-convex functions in [9].

THEOREM 4.1. *Let $f(z) \in V_k$ and let α, $0 < \alpha < 1$ be given. Then $f_\alpha(z) \in V_h$ where $h \leqq \alpha k + 2(1-\alpha)$.*

PROOF. Let $f(z)$ be given by (1.1). Then

$$(f'(\zeta))^\alpha = \exp\int_{-\pi}^{\pi} -\log(1-\zeta e^{-it})\,\alpha\, dm(t)$$

$$= \exp\int_{-\pi}^{\pi} -\log(1-\zeta e^{-it})\, d\mu(t)$$

where $d\mu(t) = \alpha\, dm(t) + (1-\alpha)\, dt/\pi$.

The second equality follows from the fact that $\int_{-\pi}^{\pi} \log(1 - ze^{-it})\,dt = 0$.

Now,

$$\int_{-\pi}^{\pi} d\mu(t) = \alpha\int_{-\pi}^{\pi} dm(t) + \frac{(1-\alpha)}{\pi}\int_{-\pi}^{\pi} dt = 2$$

and

$$\int_{-\pi}^{\pi} |d\mu(t)| \leqq \alpha\int_{-\pi}^{\pi} |dm(t)| + \frac{1-\alpha}{\pi}\int_{-\pi}^{\pi} dt \leqq \alpha k + 2(1-\alpha).$$

Thus, by Definition 1.1, $f(z) \in V_h$. Even if $f(z)$ is not univalent, $f_\alpha(z)$ will be univalent provided $\alpha < 2/(k-2)$.

5. For the case $k = 2$, i.e. convex functions, we can consider the problem mentioned in the previous section in greater detail. We begin by defining classes of functions which generalize convex functions of order β.

DEFINITION 5.1. For fixed real numbers β, γ with $0 \leqq \beta < 1$ and $-\pi/2 < \gamma < \pi/2$, $C_\beta(\gamma)$ is the class of all functions $f(z)$ which satisfy

$$\operatorname{Re} e^{i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \tag{5.1}$$

for all z in D.

We assume in the sequel that $\cos\gamma > \beta$ since otherwise $C_\beta(\gamma)$ is empty.

For $\gamma = 0$, these are the classes of convex functions of order β, which we denote by C_β. For $\beta = 0$, these are the classes of functions introduced by M. S. Robertson [8], for which $zf'(z)$ is γ-spiral. These functions are not necessarily univalent in D[8]. However, we do have

LEMMA 5.1. *The radius of convexity for* $C_\beta(\gamma)$ *is* $((\cos\gamma - \beta) + \sqrt{\beta^2 + \sin^2\gamma})^{-1}$.

The proof follows that of Robertson in [6].

PROOF. From the defining condition (5.1) we have that

$$1 + \frac{zf''(z)}{f'(z)} = e^{-i\gamma}\left[(\cos\gamma - \beta)P(z) + i\sin\gamma + \beta\right]$$

where $P(z) \in \boldsymbol{P}$. Setting $P(z) = u(z) + iv(z)$, we find the real part of this to be

$$(\cos\gamma - \beta)\left[u(z)\cos\gamma + v(z)\sin\gamma\right] + \sin^2\gamma + \beta\cos\gamma.$$

Robertson has shown [6] that if $u(z) + iv(z) \in \boldsymbol{P}$, then

$$u(z)\cos\gamma + v(z)\sin\gamma \geqq \frac{(1 + |z|^2)\cos\gamma - 2|z|}{1 - |z|^2}.$$

Thus,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqq \left[1 + |z|(2\beta - 2\cos\gamma) + |z|^2(\cos^2\gamma - \sin^2\gamma - 2\beta\cos\gamma)\right] \left[1 - |z|^2\right]^{-1}.$$

This is positive provided that

$$|z| < \left[(\cos\gamma - \beta) + \sqrt{\beta^2 + \sin^2\gamma}\right]^{-1}.$$

We now consider the problem discussed in the previous section, but now we allow α to be complex.

THEOREM 5.1. *Let* $\alpha = \rho e^{i\sigma}$ *be a given number with* $-\pi/2 < \gamma - \sigma < \pi/2$, $0 \leqq \rho < 1$. *Then,* $f(z) \in C_\beta(\gamma)$ *iff*

$$f_\alpha(z) \in C_\delta(\gamma - \sigma) \text{ where } \delta = \cos(\gamma - \sigma) - \rho \cos \gamma + \rho\beta.$$

PROOF. Recall that $f_\alpha(z) = \int_0^z (f'(\zeta))^\alpha d\zeta$. Thus, we have $f'_\alpha(z) = (f'(z))^\alpha$, $f''_\alpha(z) = \alpha(f'(z))^{\alpha-1} \cdot f''(z)$, and

$$1 + \frac{zf''_\alpha(z)}{f'_\alpha(z)} = \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)$$

Setting $\alpha = \rho e^{i\sigma}$, we find that

$$\frac{e^{i(\gamma-\sigma)}}{\rho}\left(1 + \frac{zf''_\alpha(z)}{f'_\alpha(z)}\right) = e^{i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{e^{i(\gamma-\sigma)}}{\rho}(1 - \rho e^{i\sigma})$$

or

$$\mathrm{Re}\left\{e^{i(\gamma-\sigma)}\left(1 + \frac{zf''_\alpha(z)}{f'_\alpha(z)}\right)\right\} = \rho\,\mathrm{Re}\left\{e^{i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} + \cos(\gamma - \sigma) - \rho\cos\gamma$$

Since $f(z) \in C_\beta(\gamma)$, we conclude that this last expression is

$$\geqq \rho\beta + \cos(\gamma - \sigma) - \rho\cos\gamma = \delta$$

Thus, by Definition 5.1, $f_\alpha(z) \in C_\delta(\gamma - \sigma)$. The conditions $-\pi/2 < \gamma - \sigma < \pi/2$ and $0 \leqq \rho < 1$ are to insure that $0 \leqq \delta < 1$ which is the case that interests us.

To prove the converse, we note that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{\alpha}\left(1 + \frac{zf''_\alpha(z)}{f'_\alpha(z)}\right) - \frac{1-\alpha}{\alpha}.$$

Now, since $f_\alpha(z) \in C_\delta(\gamma - \sigma)$ we conclude that

$$\mathrm{Re}\left\{e^{i\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \geqq \frac{1}{\rho}\delta - \frac{\cos(\gamma-\sigma)}{\rho} + \cos\gamma = \beta.$$

and again by Definition 5.1 $f(z) \in C_\beta(\gamma)$.

In the following three corollaries, α and β satisfy the condition of Definition 5.1.

COROLLARY 5.1. *Let* α *be real. Then* $f(z) \in C$ *iff* $f_\alpha(z) \in C_{1-\alpha}$.

PROOF. Set $\sigma = \gamma = \beta = 0$ and $\rho = \alpha$ in Theorem 5.1.

COROLLARY 5.2. *$f(z) \in C_\beta$ iff $f_\alpha \in C_{1-\alpha(1-\beta)}$,*

PROOF. Set $\sigma = \gamma = 0$ and $\rho = \alpha$ in Theorem 5.1.

COROLLARY 5.3. *Let $f \in C_\beta(\gamma)$, and let $\alpha = \rho e^{i\sigma}$. Then $f_\alpha(z)$ is convex for*

$$|z| < \left(\rho\cos\gamma - \beta + \sqrt{1 + 2\rho\cos(\gamma-\sigma)(\beta-\cos\sigma) + \rho^2(\beta-\cos\gamma)^2}\right)^{-1}$$

This follows from Lemma 5.1.

In view of the relation (1.3), we can restate Corollary 5.1 in the form of a theorem which greatly improves two theorems of A. Schild [10].

THEOREM 5.2. *For all real α, $0 < \alpha < 1$,*

(a) $$f(z) \in S^* \quad iff \quad g(z) = z\left(\frac{f(z)}{z}\right)^\alpha \in S_{1-\alpha}$$

(b) $$f(z) \in C \quad iff \quad g(z) = z(f'(z))^\alpha \in S_{1-\alpha}$$

Note that $S_{1-\alpha}$ denotes the class of starlike functions of order $1-\alpha$, i.e. functions which satisfy Re $\dfrac{zf'(z)}{f'(z)} > 1 - \alpha$.

6. In this section we discuss the behavior of V_k functions in the closed disc $|z| \leqq 1$ and obtain a covering theorem for functions in V_k. These results are contained, at least implicitly, in the literature, but we feel it worthwhile to state these results and indicate the proofs.

LEMMA 6.1 ([3], [11]). *Let $f(z) \in V_k$. Then $f(z)$ is continuous in the closed disc $|z| \leqq 1$ in the sense that* $\lim f(z)$ *as $z \to z_0$ exists where $|z| \leqq 1$ or $f(z) \to \infty$ as $z \to z_0$ where $|z_0| = 1$.*

LEMMA 6.2. *Let $f(z) \in V_k$ with $f(z)$ given as in (1.1). If $dm(t)$ concentrates no mass of weight greater than or equal to 1 at some fixed point t_0 in $[0, 2\pi]$, then $f(e^{it_0})$ is finite.*

PROOF. Let $dm(t) = d\alpha(t) - d\beta(t)$, and suppose that $d\alpha(t)$ concentrates mass of weight a at t_0 and that $d\beta(t)$ concentrates mass of weight b at t_0. Then $a - b < 1$ and furthermore, there exists an $\varepsilon > 0$ such that for any interval I of length less than or equal to 2ε about t_0, $\int_I d\alpha(t) \leqq c$, and $c + b < 1$.

Now,

$$
\begin{aligned}
\log\left|f'(re^{it_0})\right| &= \int_{t_0-\varepsilon}^{t_0+\varepsilon} -\log\left|1-re^{-i(t-t_0)}\right| d\alpha(t) \\
&\quad + \int_{t_0+\varepsilon}^{t_0-\varepsilon} -\log\left|1-re^{-i(t-t_0)}\right| d\alpha(t) \\
&\quad + \int_{-\pi}^{\pi} \log\left|1-re^{-i(t-t_0)}\right| d\beta(t) \\
&\leqq \int_{t_0-\varepsilon}^{t_0+\varepsilon} -\log\left|1-r\right| d\alpha(t) + \int_{t_0+}^{t_0-} -\log\left|\sin\varepsilon\right| d\alpha(t) \\
&\quad + b\log\left|1-r\right| + \int_{\substack{-\pi \\ t\neq t_0}}^{\pi} \log\left|1+r\right| d\beta(t) \\
&\leqq (b-a)\log\left|1-r\right| - \left(\frac{k}{2}+1\right)\log\left|\sin\varepsilon\right| + \left(\frac{k}{2}-1\right)\log 2 \\
&= (b-a)\log\left|1-r\right| + L.
\end{aligned}
$$

$$
L = -\left(\frac{k}{2}+1\right)\log\left|\sin\varepsilon\right| + \left(\frac{k}{2}-1\right)\log 2
$$

Exponentiating, we have

$$
\left|f'(re^{it_0})\right| \leqq \frac{L}{(1-r)^{a-b}}.
$$

Thus,

$$
\left|f(re^{it_0})\right| < \int_0^1 \left|f'(re^{it_0})\right| dr \leqq L \int_0^1 \frac{1}{(1-r)^{a-b}} dr = \frac{L}{1-(a-b)}
$$

and this is finite since $a-b<1$.

Lemma 6.2 is proved in [2] for $k=2$ and our proof is based on that proof. The lemmas are stated essentially as they appear in [11].

THEOREM 6.1. *Let $f(z)\in V_k$. Then, either*

a) *$f(z)$ is continuous in the closed disc $|z|\leqq 1$, or*

b) *$f(z)$ is continuous in the closed disc $|z|\leqq 1$*

with the exception of at most $[k/2+1]$ points $\{z_j\}$ $j=1,\cdots,[k/2+1]$, $|z_j|=1$ for all j. At these points, $f(z)\to\infty$ as $z\to z_j$, $|z|\leqq 1$.

PROOF. The theorem follows from Lemmas 6.1 and 6.2 together with the fact that $\int_{-\pi}^{\pi} d\alpha(t) \leqq k/2+1$.

This theorem generalizes a theorem of Suffridge [11], p. 796 for convex functions. In particular, there are at most $[k/2+1]$ rays along which $|f(z)|$ is un bounded as $|z| \to 1$.

We conclude with a covering theorem for V_k.

THEOREM 6.2. *The image of D under functions in V_k contains the schlicht disc $|z| < 1/k$.*

PROOF. It is well-known that if $f(z) \in V_k$, then

$$|f'(z)| \geqq \frac{(1-|z|)^{k/2-1}}{(1+|z|)^{k/2+1}} \quad \text{for all } z \in D.$$

Let d_r denote the radius of the largest schlicht disc centered at the origin contained in the image of $|z| < r$ under $f(z)$. Then, there is a point z_0, $|z_0| = r$, such that $|f(z_0)| = d_r$. The ray from 0 to $f(z_0)$ lies entirely in the image and the inverse image of this ray is a curve C in $|z| < r$.

$$d_r = |f(z_0)| = \int_C |f'(z)|\,|dz| \geqq \int_C \frac{(1-|z|)^{k/2-1}}{(1+|z|)^{k/2+1}}|dz|$$

$$\geqq \int_0^{|z|} \frac{(1-s)^{k/2-1}}{(1+s)^{k/2+1}} ds = \frac{1}{k}\left(1 - \left(\frac{1-r}{1+r}\right)^{k/2}\right).$$

Letting $r \to 1$ we have the desired result.

This theorem is contained in Lemma 1.3 of [5] since V_k is a linearly invariant family order $k/2$. Indeed, this proof uses only that fact and is identical with the proof in [5]. However, we feel that is of interest to state such a theorem specifically for V_k.

The author wishes to thank Professor John Pfaltzgraff for several comments and references which were useful in this section.

7. We note in conclusion that U_k consists of those functions $f(z)$ which satisfy

$$\int_{-\pi}^{\pi} \left|\operatorname{Re}\left\{re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right\}\right| d\theta \leqq k\pi$$

for $r < 1$ ($z = re^{i\theta}$). Geometrically, the condition is that the total variation of the angle which the radius vector $f(re^{i\theta})$ makes with the positive real axis is bounded by $k\pi$ as z describes the circle $|z| = r$ for each $r < 1$. Thus, U_k is the class of functions with *radius rotation* bounded by $k\pi$.

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The Hebrew University of Jerusalem